

# Large $N_c$ Continuum Reduction and the Thermodynamics of QCD

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It is noted that if large  $N_c$  continuum reduction applies to an observable, then that observable is independent of temperature for all temperatures below some critical value. This fact, plus the fact that mesons and glueballs are weakly interacting at large  $N_c$  is used as the basis for a derivation of large  $N_c$  continuum reduction for the chiral condensate. The structure of this derivation is quite general and can be extended to a wide class of observables.

PACS numbers:

One of the most intriguing recent developments in QCD is the prospect of large  $N_c$  continuum reduction[1, 2, 3, 4]. Underlying this is the two decades old idea of Eguchi and Kawai (EK) that as  $N_c$ , the number of degrees of freedom in QCD, goes to infinity, the relevant spatial size in the dynamics goes to zero[5]. This was originally formulated on the lattice and it was shown that the Wilson loop was computable on a single plaquette, from which it is deduced that the free energy density is computable on a single plaquette. Unfortunately, although the EK analysis is formally correct, it has been of limited practical significance there is a phase transition separating the EK reduced theory from the physical point as one approaches the continuum limit[6]. However, recently it was suggested that while the continuum theory cannot be reduced to a point—it can be reduced to a Euclidean box of finite size (*in terms of physical units*) while keeping the exact large  $N_c$  infinite volume result[1, 2, 3, 4]. The guts of this argument is that provided the box size is larger than the critical size for which the phase transition occurs, the EK reduction works and one can simultaneously avoid the phase transition and thus connect with the continuum limit. Numerical simulations in three and four dimensions[1, 2, 3, 4] provide strong, but not compelling, evidence that this scenario is correct. The phenomena is referred to as large  $N_c$  continuum reduction.

It has also been conjectured that the same phenomena of large  $N_c$  continuum reduction occurs for the chiral condensate: namely, the chiral condensate as computed in a finite box at large  $N_c$  becomes independent of box size for boxes larger than some critical value[7]. The conjecture is motivated by consideration of random matrix theory (RMT) which is believed to become universal for the lowest modes of the Dirac operator; this presumed universal behavior is in terms of the natural scaled variable in RMT,  $z_k = \lambda_k N_c L^4 \Sigma$  where  $\sigma$  is the chiral condensate and  $\lambda_k$  is the  $k$ th eigenvalue of the Dirac operator. One expects the same universal behavior by taking  $L \rightarrow \infty$  or  $N_c \rightarrow \infty$ .

This conjecture may seem paradoxical: it is well known that spontaneous symmetry breaking cannot occur for systems of spatial extent. The resolution to this paradox is simple: a necessary (though not sufficient) condition

for spontaneous symmetry breaking is not infinite spatial size but rather an infinite number degrees of freedom (which, of course, occurs for a system of infinite spatial extent). There can be an infinite number of degrees of freedom in a system of finite spatial extent if one has an infinite number of internal degrees of freedom; this is precisely what happens for QCD as  $N_c \rightarrow \infty$ . There is numerical evidence that the chiral condensate can be computed on a finite box by an extrapolation to infinite  $N_c$ ; moreover, the evidence is consistent with the possibility that the value of the condensate is independent of the box size beyond some critical size [7]. If the continuum reduction holds, then one can do reliable large  $N_c$  simulations on relatively size boxes leading to relatively inexpensive computations. This prospect is exciting; reliable and converged lattice QCD results may be available in the near term—albeit for a colorful world.

The present letter has two main purposes: to point out that the continuum reduction has very strong implications for the phenomenology of large  $N_c$  QCD at nonzero temperature and to provide a basis in well-known large  $N_c$  phenomenology for understanding why continuum reduction must occur for the chiral condensate (and for the expectation values of a wide class of other operators). These two issues are intimately related.

First consider the phenomenological implications for large  $N_c$  QCD at  $T \neq 0$ . The central result is that any quantity which is subject to the continuum reduction at large  $N_c$  has the property that for any temperature less than some critical temperature  $T_c$ , the quantity is unchanged from its  $T = 0$  value. This result is not exact at finite  $N_c$  but any deviations from the  $T = 0$  are subleading:

$$\frac{Q(T) - Q(0)}{Q(0)} \sim 1/N_c \text{ for } T < T_c \quad (1)$$

where  $Q(T)$  is some quantity subject to large  $N_c$  continuum reduction. Thus, for example, if the chiral condensate is subject to the continuum reduction (as conjectured in ref. [7]) then in a large  $N_c$  world as the temperature of the system as increased from zero, the chiral condensate will remain unchanged from its  $T = 0$  value until the temperature reaches a critical point.

The physical basis is quite simple. The logic of large  $N_c$  continuum reduction implies that when the Euclidean space box size is larger than the critical size, the results at large  $N_c$  do not depend on the size *or shape* of the box: if all of the sides of the box are larger than the critical length ( $L_c$ ), then the quantity does not depend further on the lengths of the sides. Thus for a box with three of the sides of infinite extent and the fourth of finite size (greater than  $L_c$ ), then a quantity subject to continuum reduction does not depend on the length of this fourth side. However, Euclidean field theory with one side of finite length and standard boundary conditions (periodic for bosons, anti-periodic for fermions) *is* just finite temperature field theory with  $T = 1/L$  (where  $L$  is the length of this side). Thus, any quantity for which large  $N_c$  continuum reduction holds will be independent of temperature for  $T < T_c$  where  $T_c = 1/L_c$  up to  $1/N_c$  corrections.

This behavior is consistent with what is known about the temperature dependence of the chiral condensate. Consider the chiral condensate at fixed low temperature in the combined large  $N_c$  and chiral limits. For simplicity, consider the case where the chiral limit is taken prior to the large  $N_c$  limit; this avoids ambiguity about defining the chiral condensate by limiting the sensitivity to the ultraviolet. Moreover, with this ordering chiral perturbation theory at finite temperature can be used to compute an analytic expression for the variation of the chiral condensate with  $T$  (at zero quark mass) [8]. Up to order  $T^6$  the result is:

$$\frac{\langle \bar{q}q \rangle_T}{\langle \bar{q}q \rangle_{T=0}} = 1 - \frac{T^2}{8f_\pi^2} - \frac{T^4}{384f_\pi^4} + \frac{T^6}{288f_\pi^6} \log \left( \frac{T}{\Lambda_q} \right) + \mathcal{O}(T^8) \quad (2)$$

where  $\langle \bar{q}q \rangle_T$  is the chiral condensate at temperature  $T$ ,  $f_\pi$  is the pion decay constant (evaluated in the chiral limit) and  $\Lambda_q$  is a parameter which depends on higher-order counter terms in the chiral lagrangian. Recall the

standard  $N_c$  scaling of  $f_\pi: f_\pi \sim N_c^{1/2}$ . Using this scaling and taking temperatures of order  $N_c^0$ , and combining it with eq. (2) yields the result that

$$\lim_{N_c \rightarrow \infty} \frac{\langle \bar{q}q \rangle_T}{\langle \bar{q}q \rangle_{T=0}} = 1 + \mathcal{O}(T^8). \quad (3)$$

There is no variation of the chiral condensate with temperature in a large  $N_c$  world up to the order calculated, as is required by large  $N_c$  continuum reduction.

Equation (3) is highly suggestive and its important to understand whether it holds generally. Fortunately, it is possible to show on very general physical grounds that the chiral condensate is independent of temperature at sufficiently low temperatures in a large  $N_c$  world. Obviously, this is of phenomenological interest (at least in a large  $N_c$  world). In addition, this observation can be used as the physical basis of a demonstration that the phenomena of continuum large  $N_c$  reduction applies to the chiral condensate. Moreover, the logic underlying the derivation can be generalized to the expectation values of a wide class Lorentz scalar local operator and not just to the this chiral condensate.

The key to this general derivation is the fact that standard 't Hooft counting[9] implies that at large  $N_c$ , mesons and glueballs have masses of order  $N_c^0$  and are weakly interacting:  $V_{\text{meson-meson}} \sim N_c^{-1}$ ,  $V_{\text{meson-glueball}} \sim N_c^{-2}$  and  $V_{\text{glueball-glueball}} \sim N_c^{-2}$  where  $V$  indicates the strength of the two-body interaction (four-point function). Now consider an extended system in which the density of mesons and glueballs is of order  $N_c^0$ . In such a system the thermodynamics will be simply that of a gas non-interacting mesons and glueballs plus corrections which vanish at large  $N_c$ . Thus, at temperature of order  $N_c^0$ , the density of a given species,  $s$ , of a meson or glueball is given by the standard expression for noninteracting bosons:

$$\rho_s(T) = d_s \int \frac{d^3k}{(2\pi)^3} \rho(T, m_s, k) (1 + \mathcal{O}(N_c^{-1})) \quad \text{with} \quad \rho(T, m_s, k) \equiv \frac{1}{e^{\sqrt{k^2 + m_s^2}/T} - 1} \quad (4)$$

where  $d_s \equiv (2J_s + 1)(2I_s + 1)$  is the degeneracy factor for a species with angular momentum  $J_s$  and isospin  $I_s$ . Note in terms of  $N_c$  counting  $\rho_s(t) \sim N_c^0$ . The free energy density as measured relative to its zero temperature value is computed in a similar manner and is given by

$$\mathcal{G}_r = -T \sum_s d_s \int \frac{d^3k}{(2\pi)^3} \log \left( 1 - e^{-\sqrt{k^2 + m_s^2}/T} \right) (1 + \mathcal{O}(N_c^{-1})) \quad (5)$$

where the summation is over species of boson. The subscript  $r$  indicates that the free energy is measured relative to its  $T = 0$  value:  $\mathcal{G}_r(T) = \mathcal{G}(T) - \mathcal{G}(0)$ . Normally, by

convention one sets  $\mathcal{G}(0)$  to zero and this distinction is unimportant. However, in the present context we will differentiate with respect to an external source, such as

the quark mass, and  $\mathcal{G}(0)$  has nonvanishing dependence on these sources.

Before proceeding it is useful to recall that strength of the interaction between a meson or a glueball and a baryon is of order unity in  $N_c$  counting and thus is not weakly interacting. So, if the system has a baryon density of order unity, eqs. (4) and (5) will be spoiled. However, the baryon mass, unlike the meson or glueball masses, is order  $N_c$  [10] and hence, for temperature of order  $N_c^0$ , the density of baryons will be of order  $e^{-N_c}$  and is negligible in the large  $N_c$  limit.

Consider the chiral condensate, *i.e.*, the expectation value of  $\bar{q}q$  where both quarks are understood to be at the same space-time position. Of course, this operator is not well defined without specifying some prescription for how to regulate it at large momentum. But, as we will see for the present purpose, the details of this prescription turns out to be largely irrelevant. For the moment let us

neglect this issue and proceed formally. To simplify issues associated with renormalization it is useful to discuss the RG invariant operator  $m\bar{q}q$  (where  $m$  is the quark mass) rather than  $\bar{q}q$  itself. From the standard form of the grand partition function as a Euclidean space functional integral it is immediately apparent that

$$m\langle\bar{q}q\rangle = m \frac{T}{V} \frac{d \log(Z)}{dm} \quad (6)$$

where the system is confined to a three-space volume  $V$  (taken to infinity in the thermodynamic limit) and is of length  $\beta = 1/T$  in the time direction;  $m$  is the quark mass. Defining the free energy density in the usual way as  $\mathcal{G} = -\frac{T}{V} \log(Z)$  it is apparent that  $\langle\bar{q}q\rangle = \frac{d\mathcal{G}}{dm}$ . Combining this with eqs. (4) and (5) along with the definition of  $\mathcal{G}_r$  yields:

$$m\langle\bar{q}q\rangle_T - m\langle\bar{q}q\rangle_{T=0} = \sum_s d_s \sigma_s \int \frac{d^3k}{(2\pi)^3} \rho(T, m_s, k) \frac{\sigma_s m_s}{\sqrt{k^2 + m_s^2}} (1 + \mathcal{O}(N_c^{-1})) \quad (7)$$

where  $\sigma_s$  the sigma commutator for the species  $s$  and is defined by  $\sigma_s \equiv m \frac{dm_s}{dm}$ .

The key point for the purposes here is the  $N_c$  scaling. By conventional large  $N_c$  scaling arguments[9, 10] it is straightforward to see that

$$\begin{aligned} \sigma_s &\sim N_c^0 \text{ for } s \in \{\text{mesons}\} \\ \sigma_s &\sim N_c^{-1} \text{ for } s \in \{\text{glueballs}\} \\ \rho(T, m_s, k) &\sim N_c^0 \text{ for } T \sim N_c^0 \\ m\langle\bar{q}q\rangle_{T=0} &\sim N_c^1. \end{aligned} \quad (8)$$

Combining these scaling rules with eq. (7) yields,

$$\frac{m\langle\bar{q}q\rangle_T - m\langle\bar{q}q\rangle_{T=0}}{m\langle\bar{q}q\rangle_{T=0}} \sim \mathcal{O}(N_c^{-1}) \quad (9)$$

This is precisely of the form of eq. (1) as required by large  $N_c$  continuum reduction.

A few words about renormalization are in order. In practice the only viable way to numerically estimate the functional integral for the partition function is via the lattice and one might worry that using a lattice regularized theory might invalidate eq. (6). However, if one uses fermions which respect chiral symmetry (*eg.* overlap fermions [11]) then one can use eq. (6) to define a regulated chiral condensate in a consistent manner. As noted above the combination  $m\bar{q}q$  appears in the Lagrangian and as such has no operator mixing and no anomalous dimension. However, this does not mean the expectation value is defined unambiguously. Indeed, even for

a free field theory this quantity diverges in the ultraviolet. To render the operator finite it is typically defined at some scale  $\mu$  where fluctuations at scales above  $\mu$  are subtracted off. Provided  $\mu$  is large enough this is computable perturbatively. Moreover in the regime  $\Lambda \gg \mu \gg T$ , where  $\Lambda$  is the cutoff scale of the regularized theory, (*eg.*  $1/a$  for a lattice regularized theory) then eq. (6) can be used to compute  $m\langle\bar{q}q\rangle_T - m\langle\bar{q}q\rangle_{T=0}$  and the result is independent of the choice of  $\mu$ . The reason for this is simple—the same high momentum fluctuations are subtracted from both the finite  $T$  and  $T = 0$  cases so that their difference does not depend on the subtraction point. Thus, the numerator of eq. (9) is independent of  $\mu$  and of order  $N_c^0$ . The denominator *does* depend on  $\mu$  but for any value of  $\mu$ , it is of order  $N_c^1$ . This yields the scaling in eq. (9) for any  $\mu$ . Finally it should be noted that in the chiral limit, the ratio in eq. (9) is independent of  $\mu$ ; both the numerator and denominator have  $\mu$ -independent contributions at leading chiral order  $m$  with all  $\mu$  dependence coming in at order  $m^2$  or higher.

It is very easy to understand the origin of eq. (9). Equation (7) has a very simple interpretation. It represents the amount each hadron contributes the spatially integrated operator  $m\bar{q}q$  weighted by the density of the hadrons. Clearly the contribution from a single meson is order  $N_c^0$  as the operator is a one-body quark operator and the number of quarks in a meson is independent of  $N_c$ . Since the density of mesons is also of order unity one sees that the shift in the condensate from  $T = 0$  is of

order unity. In contrast the condensate scales with the number of colors since it effectively counts the number of active quarks species. The scaling of eq. (9) follows from this. From this qualitative discussion, it should be clear that although the preceding derivation was for the chiral condensate, the same basic feature should occur for generic local Lorentz scalar operators. The expectation value at  $T = 0$  will be altered at finite  $T$  due to the contributions from non-interacting mesons and glueballs and such contributions are characteristically smaller in  $N_c$  counting. Of course, the issues of operator renormalization for more complicated operators becomes more involved.

The temperature independence is only supposed to hold below the phase transition; if the particles don't interact one might worry that no phase transition can occur. However, as  $N_c \rightarrow \infty$  QCD has an infinite number of narrow mesons and glueball and the number of accessible states below some energy  $E$  can grow exponentially with  $E$ . If the number of accessible meson and glueball states below  $E$  goes as  $e^{-R/T_c}$ , then the total density of mesons and glueballs  $\sum_s \rho_s(T)$  will diverge as  $T \rightarrow T_c$ . This is just the Hagedorn phenomenon and its description predates QCD[12]; presumably, the QCD phase is increasing well described as a Hagedorn transition as  $N_c \rightarrow \infty$ . Of course, as the Hagedorn transition is approached, the  $1/N_c$  suppression in meson-meson interactions is overwhelmed by an enhancement due to the diverging density; one can no longer neglect interactions and the derivation presented here breaks down. Thus at or above  $T_c$ , nothing prevents the chiral condensate (or other observable) from differing from its  $T = 0$  value at large  $N_c$ .

The phenomenological demonstration that below  $T_c$  the chiral condensate (and other observables) are independent of  $T$  at large  $N_c$  can be used to derive the full result of continuum reduction. Note, that virtually nothing in the derivation depended on the spatial size being infinite. If one repeated the derivation for a the system quantized in a box with sides of length  $L_x, L_y$  and  $L_z$  and standard boundary conditions (periodic for bosons, anti-periodic for fermions), the only alterations would be in eqs. (4) and (5) for which the following substitutions would be made:

$$\int \frac{d^3k}{(2\pi)^3} \rightarrow \sum_{n_x, n_y, n_z} \frac{1}{L_x L_y L_z}$$

$$k^2 \rightarrow \left( \frac{2\pi n_x}{L_x} \right)^2 + \left( \frac{2\pi n_y}{L_y} \right)^2 + \left( \frac{2\pi n_z}{L_z} \right)^2. \quad (10)$$

The key point is that with this substitution eq. (7) continues to hold as do the scaling rules in eq. (8) and thus so does eq. (9).

Now imagine one starts with a box of infinite extent in all four directions. Labeling one of these directions as time one can use the arguments leading to eq. (9) to

reduce the length of this dimension without altering the chiral condensate (to leading order in  $1/N_c$ ) providing we do not pass through the phase transition; this is just a temperature shift. Next one can relabel the axes and denote another axis as the time direction and reduce it. This also simply becomes a temperature shift on a box which is now finite in one spatial dimension. However, as was just argued, temperature independence of the chiral condensate holds for the case of boxes with finite sides. One can similarly reduce the other two sides leading to the result that at large  $N_c$  the chiral condensate for an arbitrary size finite box is equal to that of an infinite box provided no phase transition is crossed. This is large  $N_c$  continuum reduction.

The logic underlying this derivation of large  $N_c$  continuum reduction for the chiral condensate is complimentary to that of ref. [7]. It is based only on well known phenomenology of hadronic interactions and simple features of finite-temperature field theory and thus avoids a reliance on assumptions about the applicability of random matrix theory for this context. However, the present argument is quite limited compared to that in ref. [7] in at least one critical way. In contrast to ref. [7] it provides no insight into why chiral symmetry is broken in the first place; it merely explains why the chiral condensate does not depend on box size (at large  $N_c$ ) for boxes larger than a critical size.

The author acknowledges R. Narayanan and H. Neuberger for helpful discussions. This research was begun at the "Large N QCD" workshop at the ECT\* in Trento; the author thanks the organizers of this workshop and also thanks the ECT\* for its the hospitality. This work was supported by the U.S. Department of Energy through grant DE-FG02-93ER-40762.

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